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Characterization of Solutions to the Generalized Cauchy–Riemann System

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For a generalized biaxially symmetric potential U on a semi-disk D^+, a harmonic conjugate V is defined by the generalized Cauchy–Riemann system. There is an associated boundary value theory for the Dirichlet problem. The converse to the Dirichlet problem is considered by determining the boundary functions to which U and V converge. The unique limits are hyperfunctions on the ∂D^+. In fact, the space of hyperfunctions is isomorphic to the spaces of generalized biaxially symmetric potentials and their harmonic conjugates. A representation theorem is given for U and V terms of convolutions of certain Poisson kernels with continuous functions that satisfy a growth condition on the ∂D^+.

INTRODUCTION

In the classical Dirichlet problem there is an extensive boundary-value theory concerning the Poisson and Poisson–Stieltjes integrals on the unit disk $D: x^2 + y^2 < 1$; "arbitrary" boundary values f are specified and a harmonic function U is sought for which $U \to f$ as $r \to 1^-$. The converse to the Dirichlet problem, e.g., that of finding the boundary functions to which a given harmonic function in D converges, was solved by Gelfand [8], Johnson [11], and Kothe [13]. It was shown that the boundary function exists uniquely in the space of hyperfunctions on the ∂D by constructing an isomorphism between the spaces of harmonic functions in D and hyperfunctions on the ∂D. An essential feature represents U as a series whose terms are the convolutions of the derivatives of the Poisson kernel with continuous functions on the ∂D. The uniform norms of the sequence of continuous functions grow at an assigned rate. There is a link to an earlier work of Lohwater [14] which Douglas [7] applied to describe global constraints for harmonic continuation. Recent extensions of the converse to the Dirichlet problem to the perturbed Laplace equation, generalized axially symmetric potential equation and to the converse to the initial value problem for the heat equation may be found in [12, 21, 16].

Harmonic functions in several variables with axial or biaxial symmetry model potentials for a variety of steady-state phenomena and after suitable

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transformations [2, 5, 9] are equivalent to solutions of several partial differential equations arising in applications. Poisson integral formulae correspond to the Dirichlet problems [10, 15, 17, 19]. Moreover, for a generalized biaxially symmetric potential U there is a natural concept of harmonic conjugate V and a related boundary value theory for the Dirichlet problem associated with the *generalized Cauchy–Riemann* (GCR) *system*

$$\begin{aligned} r\,\partial_r V &= -r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\,\partial_\theta U, \\ \partial_\theta V &= r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\, r\partial_r U, \\ \rho^{(\alpha,\beta)}(\theta) &:= (\sin(\theta/2))^{2\alpha+1}(\cos(\theta/2))^{2\beta+1} \end{aligned} \tag{1}$$

on the semi-disk $D^+ : D \cap \{0<\theta<\pi\}$ for fixed parameters $\alpha \geqslant \beta$ and $\beta \geqslant -\frac{1}{2}$ or $\alpha+\beta>0$ and polar coordinates (r,θ). The generalized biaxially symmetric potential and conjugate equation arise by changing dependent variables and eliminating either U or V as in the axially symmetric $(\alpha=\beta=(n-3)/2)$ case [17]. In addition, the system is related to the pseudoanalytic function theory of Bers [4] and Vekua [20] and includes the ordinary Cauchy–Riemann system so that U and V may be harmonic conjugates in the sense of analytic function theory.

We develop the converse to the Dirichlet problem for a generalized biaxially symmetric potential U and harmonic conjugate V that are viewed as classical solutions of the GCR-system on D^+. This follows by blending function-theoretic and special function methods [1, 3, 6, 9] to extend the harmonic function theory of Johnson *et al.*

BASIC FORMULAE AND PRELIMINARIES

The normalized Jacobi polynomials $R_n^{(\alpha,\beta)}(\cos\theta) := P_n^{(\alpha,\beta)}(\cos\theta)/P_n^{(\alpha,\beta)}(1)$, $n \in N := \{0, 1, 2, \ldots\}$ which extend the role of sines and cosines in the ordinary harmonic function theory are the eigenfunctions

$$\Delta_\theta^{(\alpha,\beta)} R_n^{(\alpha,\beta)} = \lambda_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}, \qquad \lambda_n^{(\alpha,\beta)} = n(n+\alpha+\beta+1)$$

of the differential operator (3)

$$\Delta_\theta^{(\alpha,\beta)} := [\rho^{(\alpha,\beta)}(\theta)]^{-1}\partial_\theta\{\rho^{(\alpha,\beta)}(\theta)\,\partial_\theta(\cdot)\}$$

and form an orthogonal set (1, 3)

$$\int_0^\pi R_k^{(\alpha,\beta)}(\cos\phi)\,R_j^{(\alpha,\beta)}(\cos\phi)\,d\mu_{\alpha\beta}(\phi) = [\omega_k^{(\alpha,\beta)}]^{-1}\,\delta_{k,j},$$

$$\omega_k^{(\alpha,\beta)} = \frac{(2k+\alpha+\beta+1)\,\Gamma(k+\alpha+\beta+1)\,\Gamma(k+\alpha+1)}{\Gamma(k+\beta+1)\,\Gamma(k+1)\,\Gamma(\alpha+1)\,\Gamma(\beta+1)},$$

that is complete in $X := L^p[0, \pi] \cup C[0, \pi]$, $2 \leqslant p \leqslant \infty$, relative to the measure

$$d\mu_{\alpha\beta}(\phi) = \rho^{(\alpha,\beta)}(\phi)\, d\phi.$$

Expand $f \in X$ as a formal Fourier–Jacobi series

$$f(\cos\theta) = \sum_{k=0}^{\infty} a_k \omega_k R_k^{(\alpha,\beta)}(\cos\theta),$$

$$a_k = a_k(f) := \int_0^{\pi} f(\cos\phi)\, R_k^{(\alpha,\beta)}(\cos\phi)\, d\mu_{\alpha\beta}(\phi), \qquad k \in N$$

and form the Abel–Poisson and conjugate Abel–Poisson means of this series by

$$A_r(f, \cos\theta) := \sum_{k=0}^{\infty} a_k \omega_k r^k R_k^{(\alpha,\beta)}(\cos\theta)$$

$$\tilde{A}_r(f, \cos\theta) := \frac{1}{\alpha+1} \sum_{k=1}^{\infty} k a_k \omega_k r^k R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) \sin(\theta/2)\cos(\theta/2).$$

It is known [3, p. 84] that if $f \in L^p[0, \pi]$, $p \geqslant 2$, then as $r \to 1^-$,

$$A_r(f, \cos\theta) \to f(\cos\theta), \qquad \tilde{A}_r(f, \cos\theta) \to \tilde{f}(\cos\theta).$$

The limit function $\tilde{f} \in L^p[0, \pi]$ is referred to as the conjugate of f. The means of the series may be rewritten as the convolutions

$$A_r(f, \cos\theta) := K_r * f_{\cos\theta} = \int_0^{\pi} K_r(\cos\theta, \cos\phi) f(\cos\phi)\, d\mu_{\alpha\beta}(\phi),$$

$$\tilde{A}_r(f, \cos\theta) := \tilde{K}_r * f_{\cos\theta} = \int_0^{\pi} \tilde{K}_r(\cos\theta, \cos\phi)\, f(\cos\phi)\, d\mu_{\alpha\beta}(\phi)$$

of f with the generalized Abel–Poisson and conjugate Abel–Poisson kernels

$$K_r(\cos\theta, \cos\phi) := \sum_{k=0}^{\infty} r^k \omega_k R_k^{(\alpha,\beta)}(\cos\theta)\, R_k^{(\alpha,\beta)}(\cos\phi),$$

$$\tilde{K}_r(\cos\theta, \cos\phi)$$
$$:= \frac{1}{\alpha+1} \sum_{k=1}^{\infty} k \omega_k r^k R_k^{(\alpha,\beta)}(\cos\theta)\, R_{k-1}^{(\alpha+1,\beta+1)}(\cos\phi) \sin(\theta/2)\cos(\theta/2).$$

Classical harmonic conjugate solutions of the GCR-system [3, p. 84] are given by the function pair

$$U(r, \cos\theta) := A_r(f, \cos\theta),$$
$$V(r, \cos\theta) := r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\,\tilde{A}_r(f, \cos\theta). \tag{2}$$

To summarize these known results as a boundary-value problem, we state

THEOREM 1. *On the semi-disk D^+, the classical solutions of the GCR-system with conjugate boundary values*

$$U(1^-, \cos\theta) = f(\cos\theta), \quad V(1^-, \cos\theta) = \rho^{(\alpha,\beta)}(\theta)\,\tilde{f}(\cos\theta)$$

in $L^p[0, \pi]$ are the harmonic conjugate function pair in Eqs. (2).

Note that the results in this paper may be harmonically continued to D/D^+ by function-theoretic methods as found in Gilbert [9].

THE CONVERSE TO THE DIRICHLET PROBLEM

A structure theorem extending those of the ordinary harmonic functions and generalized axially symmetric potentials is stated in terms of classical solutions of the GCR-system. In analogy with the harmonic function theory $(\alpha = \beta = -\frac{1}{2})$, the convolution structure is apparent. To describe the harmonic conjugate, the notion of an operator conjugate to $\Delta_\theta^{(\alpha,\beta)}$ is necessary. It is defined as

$$(\tilde{\Delta}_\theta^{(\alpha,\beta)})^n F := \sin(\theta/2)\cos(\theta/2)(\Delta_\theta^{(\alpha,\beta)})^n (F/(\sin(\theta/2)\cos(\theta/2)).$$

The uniform norm taken on continuous functions in the segment $[0, \pi]$ is referenced in the subsequent structure theorem.

THEOREM 2. *On the semi-disk D^+, the functions U and V are classical harmonic conjugate solutions of the* GCR-*system if and only if there exists a sequence $\{g_n\}$ of continuous functions on* $[0, \pi]$ *such that*

$$\lim (2n! \, \| g_n \|)^{1/n} = 0$$

and

$$U(r, \cos\theta) = \sum_{n=0}^{\infty} (\Delta_\theta^{(\alpha,\beta)})^n A_r(g_n, \cos\theta)$$

$$V(r, \cos\theta) = r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta) \sum_{n=0}^{\infty} (\tilde{\Delta}_\theta^{(\alpha,\beta)})^n \tilde{A}_r(g_n, \cos\theta)$$

for $\alpha \geqslant \beta$ and $\alpha > -\frac{1}{2}$ or $\alpha + \beta > 0$ and $\alpha > -\frac{1}{2}$.

Proof. Let the sequence $\{g_n\} \subset C[0, \pi]$ satisfy the growth condition. Expand g_n as a Fourier–Jacobi series

$$g_n(\cos\theta) = \sum_{k=0}^{\infty} a_k \omega_k R_k^{(\alpha,\beta)}(\cos\theta), \qquad n \in N$$

and form the Abel–Poisson and conjugate Abel–Poisson means. Calculations verify that for each $n \in N$, the functions

$$\begin{aligned} U_n(r, \cos\theta) &= (\Delta_\theta^{(\alpha,\beta)})^n A_r(g_n, \cos\theta), \\ V_n(r, \cos\theta) &= r^{\alpha+\beta+1} \rho^{(\alpha,\beta)}(\theta) (\tilde{\Delta}_\theta^{(\alpha,\beta)})^n \tilde{A}_r(g_n, \cos\theta), \end{aligned}$$

are harmonic conjugate solutions of the GCR-system on D^+. Thus, the formal series

$$U(r, \cos\theta) = \sum_{n=0}^{\infty} U_n(r, \cos\theta), \qquad V(r, \cos\theta) = \sum_{n=0}^{\infty} V_n(r, \cos\theta)$$

are harmonic conjugates whose convergence is to be established. Bounds for the terms are

$$|U_n(r, \cos\theta)| \leqslant C \, \|g_n\| \sum_{k=0}^{\infty} \omega_k r^k (\lambda_k^{(\alpha,\beta)})^n.$$

$$|V_n(r, \cos\theta)| \leqslant B \, \|g_n\| \sum_{k=1}^{\infty} k\omega_k r^k (\lambda_{k-1}^{(\alpha+1,\beta+1)})^n.$$

However, $\omega_k \sim O(k^{2\alpha+1})$ and $\lambda_{k-1}^{(\alpha+1,\beta+1)} = \lambda_k^{(\alpha,\beta)} - (\alpha+\beta)$ lead to

$$\max\{|U_n(r, \cos\theta)|, |V_n(r, \cos\theta)|\} \leqslant C_0 + C_1 \, \|g_n\| \sum_{k=1}^{\infty} k\omega_k (\lambda_k^{(\alpha,\beta)})^{-n} r^k.$$

Estimating the remainder $T_n(r)$ of this series shows that

$$\begin{aligned} C_2^{-1} T_n(r) &\leqslant \|g_n\| \sum_{k=n}^{\infty} k^{2\alpha+2} [k(k+\alpha+\beta+1)]^n r^k \\ &\leqslant \|g_n\| (\alpha+\beta+1)^n \sum_{k=n}^{\infty} k^{[2\alpha]+2n+3} r^k \\ &\leqslant (\alpha+\beta+1)^n \, \|g_n\| ([2\alpha]+2n+3)! \, r^{[2\alpha]+2n+3} := H_n(r). \end{aligned}$$

Indeed, for sufficiently large C_2, we have a majorant series

$$\sum_{n=0}^{\infty} H_n(r), \qquad 0 \leqslant r \leqslant \sigma < 1$$

which is convergent since $\lim(H_n(r))^{1/n} = K \lim(2n! \| g_n \|)^{1/n} = 0$. This verifies sufficiency.

The next step reverses the argument. Let U and V have the expansions

$$U(r, \cos\theta) = A_r(f, \cos\theta), \qquad V(r, \cos\theta) = r^{\alpha+\beta+1} \rho^{(\alpha,\beta)}(\theta)\, \tilde{A}_r(f, \cos\theta)$$

with Fourier–Jacobi coefficients for which $\limsup |a_k|^{1/k} \leqslant 1$. Let $\varepsilon > 0$ be given. Reasoning as in [11, 16, 21] shows that there are sequences $\{a_{k,n}\}$ and $\{a^*_{k,n}\}$ for which

$$a_k = \sum_{n=0}^{k} a_{k,n}, \qquad k a_k = \sum_{n=0}^{k} a^*_{k,n}$$

$$\max\{|a_{k,n}|, |a^*_{k,n}|\} \leqslant B(\varepsilon)\, \varepsilon^{[2\alpha]+2n+1} k^{2n}/([2\alpha] + 2n + 1)!.$$

Let us define the series

$$F_n(r, \cos\theta) := \omega_n \sum_{k > n} a_{k,n-[2\alpha]-1} r^k R_k^{(\alpha,\beta)}(\cos\theta)(-\lambda_k^{(\alpha,\beta)})^{-n},$$

$$\begin{aligned} G_n(r, \cos\theta) := {} & \frac{r^{\alpha+\beta+1} \rho^{(\alpha,\beta)}(\theta)}{\alpha+1}\, \omega_n \sum_{k > n} a^*_{k,n-[2\alpha]-1} \\ & \times r^k R_{k-1}^{(\alpha+1,\beta+1)} (\cos\theta)(-\lambda(_{k-1}^{(\alpha+1,\beta+1)})^{-n} \\ & \times \sin(\theta/2)\cos(\theta/2). \end{aligned}$$

To see that these define functions, observe that the bounds

$$\begin{aligned} & \max\{|F_n(r, \cos\theta)|, |G_n(r, \cos\theta)|\} \\ & \leqslant \frac{B(\varepsilon)\, \varepsilon^{2n+1}}{2n!}\, \omega_n \sum_{k > n-[2\alpha]-1} k^{-2([2\alpha]+1)} \end{aligned}$$

verify the uniform convergence of each series for $\alpha > -\frac{1}{2}$ so that $g_n(\cos\theta) := F_n(1, \cos\theta)$ and $G_n(1, \cos\theta)$ are continuous functions on $[0, \pi]$. Furthermore, the appraisal implies that $\limsup(2n! \| g_n \|)^{1/n} = 0$. From the earlier representation, we now find that

$$(\Delta_\theta^{(\alpha,\beta)})^n F_n(r, \cos\theta) = \omega_n \sum_{k > n} a_{k,n-[2\alpha]-1} r^k R_k^{(\alpha,\beta)}(\cos\theta),$$

$$\begin{aligned} & (\tilde{\Delta}_\theta^{(\alpha,\beta)})^n G_n(r, \cos\theta) \\ & \quad = \frac{r^{\alpha+\beta+1} \rho^{(\alpha,\beta)}(\theta)}{\alpha+1}\, \omega_n \sum_{k > n} a^*_{k,n-[2\alpha]-1} r^k R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) \\ & \quad \times \sin(\theta/2)\cos(\theta/2), \qquad n \in N. \end{aligned}$$

The proof concludes by summing these terms and then rearranging the results.

Boundary Values of the GCR-System as Hyperfunctions

The space of hyperfunctions is identified with the spaces of boundary values corresponding to the generalized potentials and their harmonic conjugates. The space H^* of hyperfunctions is the strong dual of the space H of analytic functions on $[0, \pi]$ with the topology in (13). Generalized function refers to a Schwartz distribution so that a generalized function is a hyperfunction. To characterize the hyperfunctions in a manner that is compatible with the GCR-system, consider the following generalization of [11, 16, 21].

Theorem 3. *For integer parameters* $\alpha \geqslant \beta \geqslant 0$, *the Jacobi series* $\hat{f} := \sum_{n=0}^{\infty} a_n \omega_n R_n^{(\alpha,\beta)}$ *and* $\hat{\tilde{f}} := \sum_{n=1}^{\infty} n a_n \omega_n R_n^{(\alpha,\beta)}$ *converge to hyperfunctions on* $[0, \pi]$ *if and only if* $\limsup |a_n|^{1/n} \leqslant 1$.

Proof. The reasoning follows [11, 16, 21] quid pro quo so that the proof is sketched. Consider the analytic function $\psi \in H$. Because of the relation $(\omega_n / P_n^{(\alpha,\beta)}(1))^{1/n} \sim O(1)$, the expansion theorem [18, p. 245] applies and the series

$$\psi(\cos\theta) = \sum_{n=0}^{\infty} b_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta), \qquad \limsup |b_n|^{1/n} \leqslant 1$$

converges to a function in H whose mth partial sum is ψ_m. For $\hat{f} \in H^*$, define the inner product

$$(\hat{f}, \rho^{(\alpha,\beta)}\psi) := \int_0^{\pi} f\psi\rho^{(\alpha,\beta)}\, d\phi, \qquad \alpha \geqslant \beta \geqslant 0.$$

Reasoning as in [16, 21] with $\hat{f} \in H^*$ expanded as a Jacobi series leads to

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta), \qquad \limsup |a_n|^{1/n} \leqslant 1.$$

The procedure is analogous for $\hat{\tilde{f}} \in \tilde{H}^*$, where $\tilde{H}^*$ is the space of conjugate hyperfunctions.

Conversely, given a sequence for which the limit conditions of the theorem are met, set

$$f_m(\cos\theta) = \sum_{n=0}^{m} a_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta), \qquad \tilde{f}_m(\cos\theta) = \sum_{n=1}^{m} n a_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta).$$

To show that $f_m \to f \in H^*$, it suffices to show that $\{(f_m, \psi)\}$ is convergent for $\psi \in H$ as H and H^* (and $\tilde{H}^*$) are Frechet–Montel spaces and Montel spaces are reflexive. Therefore, weak and strong convergence coincide.

Expand $\psi \in H$ as a Jacobi series. The limit

$$(\rho^{(\alpha,\beta)} f_m, \psi) \to \sum_{n=0}^{\infty} a_n b_n \omega_n, \qquad m \to \infty$$

exists since $\limsup |a_n b_n \omega_n|^{1/n} \leqslant 1$. Thus, $\rho^{(\alpha,\beta)} \hat{f} \in H^*$ as is $\hat{f} \in H^*$ when α and β are integers. The analysis for $\hat{\tilde{f}} \in \tilde{H}^*$ is similar.

This characterization identifies the space of hyperfunctions with the spaces of boundary values of the generalized biaxially symmetric potentials and their harmonic conjugates. To proceed to the boundary values, define the convolution maps $*: H^* \times H^* \to H^*$ and $\circledast: \tilde{H}^* \times \tilde{H}^* \to \tilde{H}^*$ by the Hadamard products

$$(\hat{f} * \hat{g})_{\cos\theta} := \sum_{n=0}^{\infty} a_n b_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta),$$

$$(\hat{\tilde{f}} \circledast \hat{\tilde{g}})_{\cos\theta} := \sum_{n=1}^{\infty} n a_n b_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta).$$

The spaces $\{H^*, *, +\}$ and $\{\tilde{H}^*, \circledast, +\}$ are C-rings with the identity elements

$$\hat{j}(\cos\theta) = \sum_{n=0}^{\infty} \omega_n R_n^{(\alpha,\beta)}(\cos\theta), \qquad \hat{\tilde{j}}(\cos\theta) = \sum_{n=1}^{\infty} n\omega_n R_n^{(\alpha,\beta)}(\cos\theta).$$

The Abel means are formed by $j_r(\cos\theta) = A_r(\hat{j}, \cos\theta)$ and $\tilde{j}_r(\cos\theta) = \tilde{A}_r(\hat{\tilde{j}}, \cos\theta)$ with $j_r(\cos\theta) \to \hat{j}(\cos\theta) \in H^*$ and $\tilde{j}_r(\cos\theta) \to \hat{\tilde{j}}(\cos\theta) \in \tilde{H}^*$ as $r \to 1^-$. The identification theorem follows.

THEOREM 4. *The functions U and V are classical harmonic conjugate solutions of the GCR-system on the semi-disk D^+ if and only if a unique hyperfunction $\hat{f} \in H^*$ exists such that for integers $\alpha \geqslant \beta \geqslant 0$*

$$\begin{aligned} U(r, \cos\theta) &= (j_r * \hat{f})_{\cos\theta} \\ V(r, \cos\theta) &= r^{\alpha+\beta+1} \rho^{(\alpha,\beta)}(\theta) (\tilde{j}_r \circledast \hat{\tilde{f}})_{\cos\theta}. \end{aligned} \tag{3}$$

And, $U(r, \cos\theta) \to \hat{f}(\cos\theta) \in H^*$ and $V(r, \cos\theta) \to \rho^{(\alpha,\beta)}(\theta) \hat{\tilde{f}}(\cos\theta) \in \tilde{H}^*$ as $r \to 1^-$.

Proof. Let U and V be classical harmonic conjugate solutions of the GCR-system with Fourier–Jacobi coefficients such that $\limsup |a_n|^{1/n} \leqslant 1$. Set

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta), \qquad \hat{\tilde{f}}(\cos\theta) = \sum_{n=1}^{\infty} n a_n \omega_n R_n^{(\alpha,\beta)}(\cos\theta).$$

Then, $U(r, \cos\theta) = (j_r * \hat{f})_{\cos\theta}$ and $V(r, \cos\theta) = r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)(\tilde{j}_r \circledast \hat{\tilde{f}})_{\cos\theta}$. The argument of the earlier theorem together with the Abel summability gives $U(r, \cos\theta) \to \hat{f}(\cos\theta) \in H^*$ and $V(r, \cos\theta) \to \rho^{(\alpha,\beta)}(\theta)\hat{\tilde{f}}(\cos\theta) \in \hat{\tilde{H}}^*$. And, if $\hat{f} \in H^*$ and $\hat{\tilde{f}} \in \tilde{H}^*$, Eqs. (3) are valid with $\limsup |a_n|^{1/n} \leqslant 1$. This implies that the functions

$$U(r, \cos\theta) = (j_r * \hat{f})_{\cos\theta}, \qquad V(r, \cos\theta) = r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)(\tilde{j}_r \circledast \hat{\tilde{f}})_{\cos\theta}$$

are classical harmonic conjugate solutions of the GCR-system and completes the proof.

Concluding Remarks

Because of the symmetry $R_n^{(\alpha,\beta)}(-\cos\theta) = R_n^{(\beta,\alpha)}(\cos\theta)$, equivalent theorems may be stated for the parameters reversed. Alternate forms of Theorem 1 may be derived for polynomial sequences or Radon measures as in the case for ordinary harmonic functions [11] provided that growth conditions are satisfied. Moreover, the preceding results may be extended to a broader form for generalized biaxially symmetric potentials by using generalized Abel summability and fractional differential operators as found in [3].

References

1. R. Askey, "Orthogonal Polynomials and Special Functions," Regional Conf. Series in Applied Math., SIAM, Philadelphia, 1975.
2. K. Bauer and S. Ruscheweyh, "Differential Operators for Partial Differential Equations and Function Theoretic Methods," Lecture Notes in Math., Vol. 791, Springer-Verlag, New York, 1980.
3. H. Bavinck, "Jacobi Series and Approximation," Mathematical Centre Tracts, No. 39, Mathematisch Centrum, Amsterdam, 1972.
4. L. Bers, "Theory of Pseudo-Analytic Functions," Lecture Notes, Institute for Mathematics and Mechanics, New York University, 1953.
5. L. Bragg & J. Dettman, Expansions of solutions of certain hyperbolic and elliptic problems in terms of Jacobi polynomials, *Duke J. Math.* **36** (1969), 129–144.
6. D. L. Colton, "Solution of Boundary Value Problems by the Method of Integral Operators," Research Notes in Math., Vol. 6, Pitman, San Francisco, 1976.
7. J. Douglas, Jr., Approximate continuation of harmonic and parabolic equations, *in* "Proc. of the Symposium on the Numerical Solution of Partial Differential Equations," Univ. of Maryland, pp. 353–364, 1965.
8. I. M. Gelfand and G. E. Shilov, Quelques applications de la théorie des fonctions généralisées, *J. Math. Pures Appl.* **35** (1956), 383–413.
9. R. P. Gilbert, "Function Theoretic Methods in Partial Differential Equations," Math. in Science and Engineering, Vol. 54, Academic Press, New York, 1969.
10. A. Huber, Some Results on Generalized Axially Symmetric Potentials, *in* "Proceedings, Conf. Differential Equations," Univ. of Maryland, pp. 147–155, 1955.

11. G. JOHNSON, JR., Harmonic functions in the unit disk, I and II, *Illinois J. Math.* **12** (1968), 366–396.
12. J. KELINGOS AND P. STAPLES, A Characterization of Solutions to a Perturbed Laplace Equation, *Illinois J. Math.* **22** (1978), 208–216.
13. M. KOTHE, "Topological Vector Spaces I," Springer-Verlag, New York, 1969.
14. A. J. LOHWATER, A uniqueness theorem for harmonic functions, *Proc. Amer. Math. Soc.* **3** (1952), 278–79.
15. P. A. MCCOY, H^p representation of ultraspherical expansions, *Appl. Anal.* **13** (1982), 139–148.
16. P. A. MCCOY, Converse initial value problems for a class of heat equations, *in* "Proc. Dundee Conf. on Ordinary and Partial Differential Equations," 1982, Lecture Notes in Math., Vol. 964, (W. N. Everitt and B. D. Sleeman Eds.), Springer-Verlag, pp. 494–503, 1982.
17. B. MUCKENHOUPT AND E. M. STEIN, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17–91.
18. G. SZEGO, "Orthogonal Polynomials," Amer. Math. Soc. Publications, vol. 23, 3rd ed., Providence, R. I., 1967.
19. R. WEINACHT, N. HALL, AND D. QUINN, Poisson integral formulas in generalized bi-axially symmetric potential theory, *SIAM J. Math. Anal.* (1974), 111–118.
20. I. VEKUA, "Generalized Analytic Functions," Addison–Wesley, Reading, Mass., (1962).
21. A. ZAYED, Hyperfunctions as Boundary Values of Generalized Axially Symmetric Potentials, *Illinois J. Math.* **25** (1981), 306–317.